# THE ASKEY-WILSON FUNCTION TRANSFORM SCHEME

Erik Koelink and Jasper V. Stokman

December 23, 1999

Abstract. In this paper we present an addition to Askey's scheme of $q$-hypergeometric orthogonal polynomials involving classes of $q$-special functions which do not consist of polynomials only. The special functions are $q$-analogues of the Jacobi and Bessel function. The generalised orthogonality relations and the second order $q$-difference equations for these families are given. Limit transitions between these families are discussed. The quantum group theoretic interpretations are discussed shortly.

## 1. Introduction

The Askey-scheme of hypergeometric orthogonal polynomials is a scheme containing various known sets of orthogonal polynomials that can be written in terms of hypergeometric series, see e.g. Askey and Wilson [2], Koekoek and Swarttouw [14], Koornwinder [27]. A typical entry is the set of Jacobi polynomials defined by

$$R_n^{(\alpha,\beta)}(x) = {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right), \qquad n \in \mathbb{Z}_{\geq 0}, \tag{1.1}$$

where we use the standard notation for hypergeometric series, see e.g. [7]. The Jacobi polynomials are orthogonal with respect to the beta distribution $(1-x)^\alpha(1+x)^\beta$ on the interval $[-1,1]$. Moreover, we have the much bigger $q$-analogue of the Askey-scheme having the Askey-Wilson polynomials and Racah polynomials at the top level with four degrees of freedom (apart from $q$), see e.g. [14].

The relation with schemes of non-polynomial special functions are less well-advertised, and here we discuss the $q$-analogue of the scheme displayed in Figure 1.1. The Jacobi function transform is an integral transform on $[0,\infty)$ in which the kernel is given by a Jacobi function

$$\phi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1\left(\begin{matrix} \frac{1}{2}(\alpha+\beta+1-i\lambda), \frac{1}{2}(\alpha+\beta+1+i\lambda) \\ \alpha+1 \end{matrix}; -\sinh^2 t\right) \tag{1.2}$$

1991 *Mathematics Subject Classification.* 33D15, 33D45 (Primary) 33D80 (Secondary).

*Key words and phrases.* Askey-Wilson function, big $q$-Jacobi function, little $q$-Jacobi function, $q$-Bessel function, $q$-Askey scheme, difference operator, limit transitions, quantum groups.

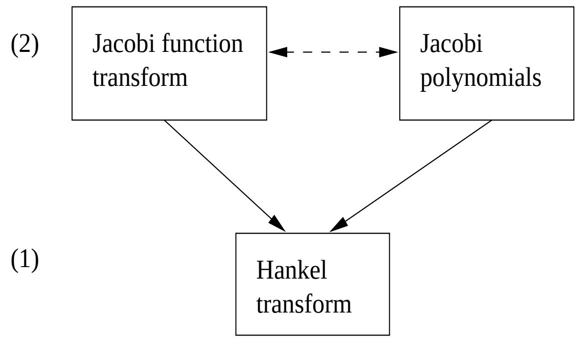


FIGURE 1.1. Jacobi function scheme.

for $|\sinh t| < 1$, which has a one-valued analytic continuation to $-\sinh^2 t \in \mathbb{C}\backslash[1,\infty)$, see Koornwinder's survey paper [24] for a nice introduction and references. We see that we may view the Jacobi function as an analytic continuation in its degree of the Jacobi polynomial; replace $n$ in (1.1) by $\frac{1}{2}(i\lambda - \alpha - \beta - 1)$ and $x$ by $\cosh 2t$ to get the Jacobi function of (1.2).

The Jacobi function transform is then given by

$$\begin{aligned} g(\lambda) &= \int_0^\infty f(x)\phi_\lambda^{(\alpha,\beta)}(t)(2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}\,dt, \\ f(t) &= \int_0^\infty g(\lambda)\phi_\lambda^{(\alpha,\beta)}(t)\frac{|\Gamma(\frac{1}{2}(i\lambda+\alpha+\beta+1))\Gamma(\frac{1}{2}(i\lambda+\alpha-\beta+1))|^2}{4^{\alpha+\beta+1}|\Gamma(\alpha+1)\Gamma(i\lambda)|^2}\,d\lambda \end{aligned} \tag{1.3}$$

for some suitable class of functions. Here we assume that $\alpha,\beta\in\mathbb{R}$ satisfy $|\beta| < \alpha+1$, otherwise discrete mass points have to be added to the Plancherel measure, see [24, §2].

The Hankel transform is the integral transform on $[0,\infty)$ that has the Bessel function

$$J_\alpha(x) = \frac{(x/2)^\alpha}{\Gamma(\alpha+1)}\,{}_0F_1\left(\begin{matrix}-\\ \alpha+1\end{matrix};-\frac{x^2}{4}\right) \tag{1.4}$$

as its kernel. For suitable functions and $\alpha > -1$ the Hankel transform pair is given by, see Watson [39, §14.3],

$$g(\lambda) = \int_0^\infty f(x)\,J_\alpha(x\lambda)x\,dx, \qquad f(x) = \int_0^\infty g(\lambda)\,J_\alpha(x\lambda)\lambda\,d\lambda. \tag{1.5}$$

The Hankel transform can formally be obtained as a limit case from the orthogonality relations for the Jacobi polynomials using the limit

$$\lim_{N\to\infty} R_{n_N}^{(\alpha,\beta)}(1-\frac{x^2}{2N^2}) = {}_0F_1(-;\alpha+1;-(x\lambda)^2/4), \quad n_N/N\to\lambda \text{ as } N\to\infty. \tag{1.6}$$

The Hankel transform can also be viewed as a limit case of the Jacobi function transform by use of the limit transition

$$\lim_{\varepsilon\downarrow 0}\phi_{\lambda/\varepsilon}^{(\alpha,\beta)}(t\varepsilon) = 2^\alpha\Gamma(\alpha+1)(\lambda t)^{-\alpha}\,J_\alpha(\lambda t), \tag{1.7}$$

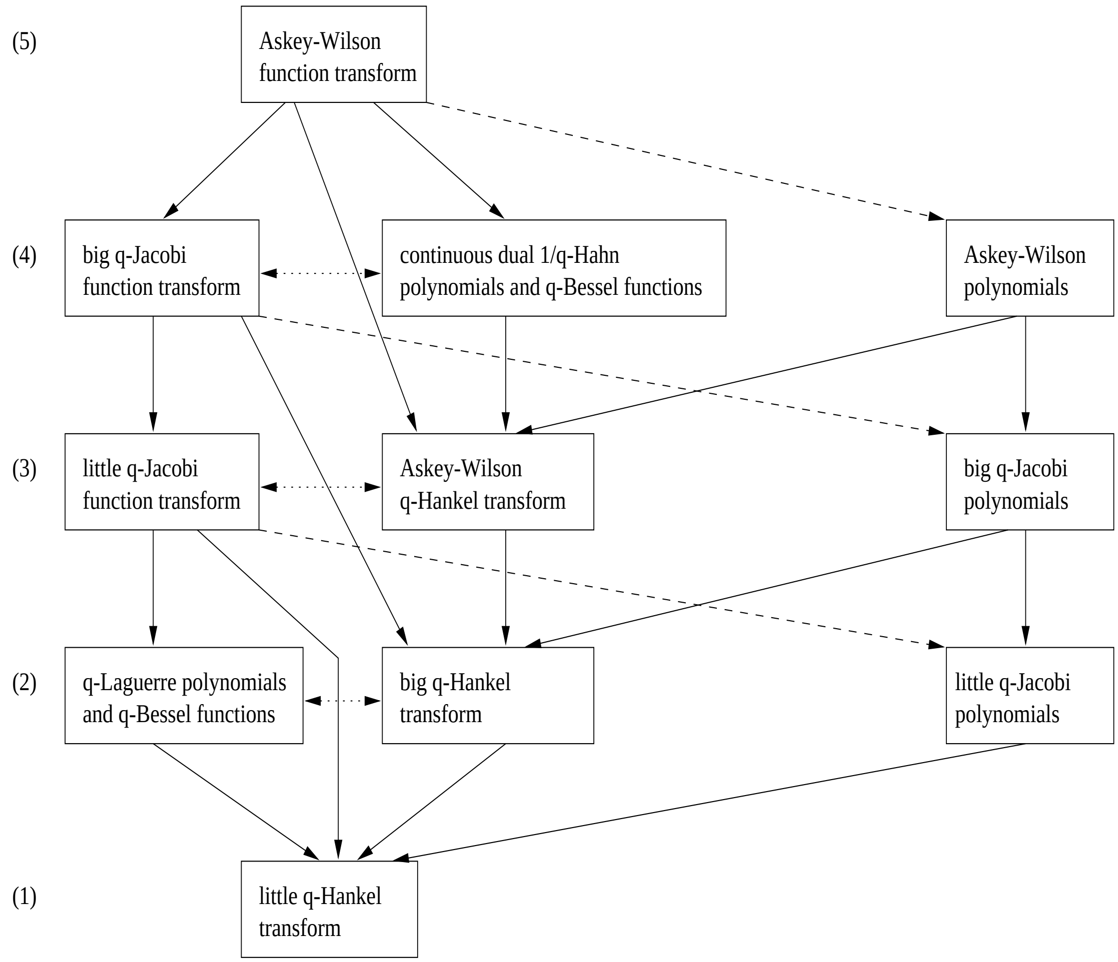


FIGURE 1.2. Askey-Wilson function scheme.

see [24, (2.34)], and the limit transition (1.7) can be considered as a generalisation of (1.6).

So Figure 1.1 gives an addition to the Askey-scheme of hypergeometric orthogonal polynomials by an analytic continuation in the spectral parameter (the dashed line) to the Jacobi functions, and by a limit transition to the Bessel functions. The Jacobi functions and polynomials have two degrees of freedom, namely $\alpha$ and $\beta$, and the Hankel transform has one degree of freedom, namely the order $\alpha$ of the Bessel function. We consider the Jacobi functions as the master functions, and the Jacobi polynomials and the Bessel functions as derivable functions. We discuss the group theoretic interpretation of Figure 1.1 in §7.1.

The purpose of this paper is to give three $q$-analogues of Figure 1.1; one related to the Askey-Wilson polynomials, presented in §2, one related to the big $q$-Jacobi polynomials, presented in §3, and one related to the little $q$-Jacobi polynomials, presented in §4. This is depicted in Figure 1.2, where the three boxes on the right hand side are part of Askey's scheme of $q$-hypergeometric orthogonal polynomials, see [14]. These three $q$-analogues are related by limit transitions as well, and this is also depicted in Figure 1.2. This is discussed in §5 and §6, where the boxes with the continuous dual $q^{-1}$-Hahn polynomials and the $q$-Laguerre polynomials are discussed.

In Figure 1.2 the lines denote limit transitions, the dashed lines denote analytic continuation and the dotted lines, which do not appear in Figure 1.1, denote that the results of the two boxes involved are related by duality. In particular, we think of the Askey-Wilson function transform and the little $q$-Hankel transform as self-dual transforms, i.e. the inverse transform equals the generic Fourier transform itself, possibly for dual parameters. Note that there is a striking difference between Figure 1.1 and Figure 1.2. The dashed line corresponding to analytic continuation in Figure 1.1 is horizontal, i.e. the Jacobi polynomials and the Jacobi function transform both have 2 degrees of freedom. The dashed lines in Figure 1.2 go down, i.e. the $q$-analogue of the Jacobi function transform has one extra degree of freedom compared to the $q$-analogue of the Jacobi polynomial. This extra parameter is not contained in the definition of the $q$-analogues of the Jacobi function, but it appears in the measure for the $q$-analogue of the Jacobi function transform.

In §7 we discuss the (quantum) group theoretic interpretation of Figure 1.1 and Figure 1.2, which is also the motivation for the names for these transforms. We end with some concluding remarks and some open problems.

Let us finally note that this paper does not contain rigorous proofs. The transform pairs are obtained by a spectral analysis of a second order $q$-difference operator that is symmetric on a suitable weighted $L^2$-space, see the references in §§2, 3 and 4. The limit transitions are only considered on a formal level and are emphasised via the second order $q$-difference operators involved.

*Notation.* We use the notation for basic hypergeometric series as in the book [7] by Gasper and Rahman. We assume throughout $0 < q < 1$.

*Acknowledgement.* The second author is supported by a NWO-Talent stipendium of the Netherlands Organization for Scientific Research (NWO). Part of the research was done while the second author was supported by the EC TMR network "Algebraic Lie Representations", grant no. ERB FMRX-CT97-0100.

## 2. Askey-Wilson analogue of the Jacobi function scheme

In this section we consider the analogue of Figure 1.1 on the level for the Askey-Wilson case. So the second order difference operator is

$$Lf(x) = A(x)\big(f(qx) - f(x)\big) + B(x)\big(f(q^{-1}x) - f(x)\big), \tag{2.1}$$

where

$$A(x) = \frac{(1-ax)(1-bx)(1-cx)(1-dx)}{(1-x^2)(1-qx^2)}, \qquad B(x) = A(x^{-1}). \tag{2.2}$$

The difference operator (2.1), (2.2) has been introduced by Askey and Wilson [2]. The general set of eigenfunctions for (2.1) with $A$ and $B$ as in (2.2) is given by Ismail and Rahman [11].

**2.1. The Askey-Wilson functions.** The Askey-Wilson functions are defined by

$$\begin{aligned}(2.3)\quad \phi_\gamma(x;a;b,c;d|q) = &\frac{\big(qax\gamma/\tilde{d}, qa\gamma/\tilde{d}x;q\big)_\infty}{\big(\tilde{a}\tilde{b}\tilde{c}\gamma, q\gamma/\tilde{d}, q\tilde{a}/\tilde{d}, qx/d, q/dx;q\big)_\infty} \\ &\times {}_8W_7\left(\tilde{a}\tilde{b}\tilde{c}\gamma/q; ax, a/x, \tilde{a}\gamma, \tilde{b}\gamma, \tilde{c}\gamma; q, q/\tilde{d}\gamma\right)\end{aligned}$$

for $|\gamma| > |q/\tilde{d}|$, where $\tilde{a} = \sqrt{q^{-1}abcd}$, $\tilde{b} = ab/\tilde{a}$, $\tilde{c} = ac/\tilde{a}$, and $\tilde{d} = ad/\tilde{a}$. Observe that the Askey-Wilson function in (2.3) is symmetric in $b$ and $c$, and almost symmetric in $a$ and $b$;

$$(2.4)\qquad \phi_\gamma(x;a;b,c;d|q) = \frac{(\tilde{c}\gamma, qb/d, \tilde{c}/\gamma;q)_\infty}{(q\gamma/\tilde{d}, qa/d, q/\tilde{d}\gamma;q)_\infty}\phi_\gamma(x;b;a,c;d|q),$$

by an application of [7, (III.36)], or see Suslov [36]. Then $L\phi_\gamma = \big(-1-\tilde{a}^2+\tilde{a}(\gamma+\gamma^{-1})\big)\phi_\gamma$. Note that the invariance $x \leftrightarrow x^{-1}$ is obvious in (2.3). There exists a meromorphic continuation of the Askey-Wilson function in $\gamma$ which is invariant under $\gamma \leftrightarrow \gamma^{-1}$ by [7, (III.23)]. By [7, (III.23)] again we see that the Askey-Wilson function is self-dual in the sense that

$$(2.5)\qquad \phi_\gamma(x;a;b,c;d|q) = \phi_x(\gamma;\tilde{a};\tilde{b},\tilde{c};\tilde{d}|q).$$

Assume that (i) $0 < b, c \le a < d/q$, (ii) $bd, cd \ge q$ and (iii) $ab, ac < 1$, and let $t < 0$ be an extra parameter. By $\mathcal{H}(a;b,c;d;t)$ we denote the weighted $L^2$-space of symmetric functions $f(x) = f(x^{-1})$ with respect to the measure $d\nu(x;a,b,c;d|q,t)$, which is defined as follows. We introduce

$$(2.6)\qquad \Delta(x) = \frac{(x^{\pm 2}, qx^{\pm 1}/d;q)_\infty}{\theta(tdx^{\pm 1})\,(ax^{\pm 1}, bx^{\pm 1}, cx^{\pm 1};q)_\infty},$$

where $\theta(x) = (x, q/x;q)_\infty$ is the (renormalised) Jacobi theta-function and where we use $(cx^{\pm 1};q)_\infty = (cx, c/x;q)_\infty$ and similarly for the other $\pm$-signs. Note that this the standard Askey-Wilson weight function, see [2], [7, Ch. 6], multiplied by $q$-constant function that can be written as a quotient of theta-functions. The positive measure is now given by

$$(2.7)\qquad \int_{\mathbb{C}^*} f(x)d\nu(x;a;b,c;d|q,t) = \frac{K}{4\pi i}\int_{\mathbb{T}} f(x)\Delta(x)\frac{dx}{x} + K\sum_{s\in S} f(s)\mathrm{Res}_{x=s}\frac{\Delta(x)}{x},$$

where $S = S_+ \cup S_-$, $S_+ = \{aq^k \mid k \in \mathbb{Z}_{\ge 0},\, aq^k > 1\}$ and $S_- = \{tdq^k \mid k \in \mathbb{Z},\, tdq^k < 1\}$, under the generic assumption on the parameters that $S \cup S^{-1}$ consists of simple poles of $\Delta$. Note that this condition can be removed by extending the definition of the masses at the discrete set by continuity, see [23] for details. Here $K = K(a;b,c;d;t)$ is a positive constant defined by

$$(2.8)\qquad K = (qabcdt^2)^{-\frac{1}{2}}(ab, ac, bc, qa/d, q;q)_\infty\big(\theta(qt)\theta(adt)\theta(bdt)\theta(cdt)\big)^{\frac{1}{2}}.$$

The Askey-Wilson function transform pair for a sufficiently nice function $u \in \mathcal{H}(a;b,c;d;t)$ is given by

$$
\begin{aligned}
\hat{u}(\gamma) &= \int_{\mathbb{C}^*} u(x)\phi_\gamma(x;a;b,c;d|q)\, d\nu(x;a;b,c;d|q,t), \\
u(x) &= \int_{\mathbb{C}^*} \hat{u}(\gamma)\phi_\gamma(x;a;b,c;d|q)\, d\nu(\gamma;\tilde{a};\tilde{b},\tilde{c};\tilde{d}|q,\tilde{t}),
\end{aligned} \tag{2.9}
$$

where $\tilde{t} = 1/qadt$. Furthermore, $(\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{t})$ satisfy the same conditions as $(a,b,c,d,t)$. So, in view of (2.5), the inverse of the Askey-Wilson function transform is the Askey-Wilson transform for the dual set of parameters. Moreover, the Askey-Wilson function transform extends to an isometric isomorphism from $\mathcal{H}(a,b,c;d;t)$ to $\mathcal{H}(\tilde{a},\tilde{b},\tilde{c};\tilde{d};\tilde{t})$. Proofs of these results can be found in [23]. See Suslov [35], [36] for Fourier-Bessel type orthogonality relations for the Askey-Wilson functions.

**2.2. The Askey-Wilson polynomials.** The Askey-Wilson polynomials are the eigenfunctions of $L$ as in (2.1), (2.2), which are polynomial in $\frac{1}{2}(x+x^{-1})$. These orthogonal polynomials are very well-known, see [2], [7, §7.5], and they are on top of the Askey scheme of basic hypergeometric orthogonal polynomials, see [14]. See also Brown, Evans and Ismail [4] for an operator approach to the Askey-Wilson polynomials and their orthogonality relations. The Askey-Wilson functions reduce to the Askey-Wilson polynomials for $\gamma^{-1} = \tilde{a}q^n$, $n \in \mathbb{Z}_{\geq 0}$, since the ${}_8W_7$-series in (390) reduces to a terminating series

$$
\begin{aligned}
&{}_8W_7(aq^{-n}/d; ax, a/x, q^{-n}, q^{1-n}/cd, q^{1-n}/bd; q, q^n bc) = \\
&\qquad \frac{(aq^{1-n}/d, q^{1-n}/ad;q)_n}{(q^{1-n}/dx, q^{1-n}x/d;q)_n}\, {}_4\varphi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ax, a/x \\ ab, ac, ad \end{matrix}; q, q\right)
\end{aligned} \tag{2.10}
$$

by [7, (III.18)]. We can also use [7, (III.36)] to write the ${}_8W_7$-series as a sum of two balanced ${}_4\varphi_3$-series, which reduces to a single terminating balanced ${}_4\varphi_3$-series for $\gamma^{-1} = \tilde{a}q^n$, see e.g. Suslov [35], [36]. This shows that the Askey-Wilson function $\phi_\gamma(x;a;b,c;d|q)$ is the analytic continuation of the Askey-Wilson polynomial.

**2.3. The Askey-Wilson $q$-Bessel functions.** In (2.3) we replace $c$ by $c\varepsilon$, $d$ by $d/\varepsilon$ and $\gamma$ by $\gamma\varepsilon$, then the formal limit transition gives

$$
\lim_{\varepsilon \downarrow 0} \phi_{\gamma\varepsilon}(x;a;b,c\varepsilon;\frac{d}{\varepsilon}|q) = {}_2\varphi_1\left(\begin{matrix} ax, a/x \\ ab \end{matrix}; q, \frac{q}{\tilde{d}\gamma}\right). \tag{2.11}
$$

Taking the limit in the second order $q$-difference equation shows that the Askey-Wilson $q$-Bessel function

$$
J_\gamma(x;a;b|q) = {}_2\varphi_1\left(\begin{matrix} ax, a/x \\ ab \end{matrix}; q, -\frac{q\gamma}{a}\right) \tag{2.12}
$$

is a solution to $LJ_\gamma(\cdot;a;b|q) = \gamma J_\gamma(\cdot;a;b|q)$, with $L$ as in (2.1) with

$$
A(x) = \frac{(1-ax)(1-bx)x}{(1-x^2)(1-qx^2)}, \qquad B(x) = A(x^{-1}). \tag{2.13}
$$

Taking the limit transition (2.11) through the sequence $\varepsilon = q^m$, $m \to \infty$, in the Askey-Wilson function transform pair with $t$ replaced $\varepsilon t$ formally leads to the following orthogonality relations

$$
\begin{aligned}
(2.14)\qquad & \int_{\mathbb{C}^*} J_{\gamma q^k}(x;a;b|q)\, J_{\gamma q^l}(x;a;b|q)\, d\nu(x;a;b,q\gamma;\gamma^{-1}|q,-1) = \\
& \qquad \delta_{k,l} a^{-2k} \frac{(-aq^{-k}/\gamma;q)_\infty}{(-bq^{-k}/\gamma;q)_\infty} \frac{K(a;b,q\gamma;\gamma^{-1};-1)}{\big((ab;q)_\infty \theta(-a/\gamma)\big)^2}, \qquad k,l \in \mathbb{Z},
\end{aligned}
$$

where $\gamma = -a\tilde{t}$ for $a, b, \gamma > 0$, $ab < 1$, $a > b$ and $K$ as in (2.8). Observe that cancellation in the weight $\Delta$ (2.6) occurs in (2.14), since $cd = q$. The orthogonality relations can be obtained from Kakehi [12], see also [22, App. A] where also other ranges of the parameters are considered. Moreover, the functions $J_{\gamma q^k}(\cdot;a;b|q)$ form a complete set in the weighted $L^2$-space for the measure in (2.14). Note that under the limit transition (2.11) only the infinite set discrete mass points that tend to $-\infty$ in (2.9) survive.

Using the same limit transition (2.11) in (2.10) we find the Askey-Wilson $q$-Bessel functions (2.12) with the corresponding orthogonality relations (2.14) from the orthogonality relations of the Askey-Wilson polynomials, see also [15, Ch. 3]. The Askey-Wilson $q$-Bessel function is also studied by Bustoz and Suslov [5], who derive Fourier series expansions, and by Ismail, Masson and Suslov [10], who derive Fourier-Bessel type orthogonality relations for $J_\gamma(x;a;b|q)$.

## 3. Big $q$-analogue of the Jacobi function scheme

In this section we consider the analogue of Figure 1.1 on the level of the big $q$-Jacobi case. So we consider the second order difference operator $L$ as in (2.1) with

$$
(3.1)\qquad A(x) = a^2(1+\frac{1}{abx})(1+\frac{1}{acx}), \qquad B(x) = (1+\frac{q}{bcx})(1+\frac{1}{x}).
$$

The general set of eigenfunctions for $L$ in (2.1) with $A$ and $B$ as in (3.1) is given by Gupta, Ismail and Masson [8].

**3.1. The big $q$-Jacobi functions.** The big $q$-Jacobi functions are defined by

$$
(3.2)\qquad \phi_\gamma(x;a;b,c;q) = {}_3\varphi_2\left(\begin{matrix} a\gamma, a/\gamma, -1/x \\ ab, ac \end{matrix}; q, -bcx\right), \qquad |bcx| < 1,
$$

and they satisfy $L\phi_\gamma(\cdot;a;b,c;q) = (-1-a^2+a(\gamma+\gamma^{-1}))\phi_\gamma(\cdot;a;b,c;q)$. We can extend the definition of the big $q$-Jacobi function (3.2) to generic values of $x$ by requiring that this second order $q$-difference equation remains valid, see [21]. Assume that the parameters $a$, $b$ and $c$ are positive, $a$ greater than $b$ and $c$, and that all pairwise products are less than one, and let $z > 0$. The big $q$-Jacobi function transform is given by the following transform

pair

$$
\begin{aligned}
\hat{u}(\gamma) &= \int_{-1}^{\infty(z)} u(x)\phi_\gamma(x;a;b,c;q)\frac{(-qx,-bcx;q)_\infty}{(-abx,-acx;q)_\infty}\,d_qx,\\
u(x) &= C\int_{\mathbb{C}^*}\hat{u}(\gamma)\,\phi_\gamma(x;a;b,c;q)\,(\gamma^{\pm1}abc;q)_\infty^{-1}\,d\nu(\gamma;a;b,c;q/abc|q,-z^{-1}),\\
C &= \frac{\theta(-abz)\theta(-acz)\theta(-bcz)\,(ab,ac;q)_\infty^2}{(1-q)z\,\theta(-qz)K(a;b,c;q/abc;-1/z)}
\end{aligned}
\tag{3.3}
$$

with the notation of (2.7) and where the $q$-integral is defined by

$$
\int_{-1}^{\infty(z)} f(x)\,d_qx = (1-q)\sum_{k=0}^{\infty} f(-q^k)q^k + (1-q)z\sum_{k=-\infty}^{\infty} f(zq^k)q^k.
$$

After a suitable scaling the big $q$-Jacobi function transform extends to an isometric isomorphism of the weighted $L^2$-space with respect to the $q$-integral with weight as in the first equality of (3.3) onto the weighted $L^2$-space of symmetric functions with respect to the weight $C(\gamma^{\pm1}abc;q)_\infty^{-1}\,d\nu(\gamma;a,b,c;q/abc|q,-1/z)$. See [21] for a proof of these statements.

**3.2. The big $q$-Jacobi polynomials.** The polynomial eigenfunctions to (2.1), (3.1) are the big $q$-Jacobi polynomials and they are given by the big $q$-Jacobi functions (3.2) with $\gamma = aq^n$, $n\in\mathbb{Z}_{\geq 0}$,

$$
{}_3\varphi_2\left(\begin{matrix} a^2q^n, q^{-n}, -1/x \\ ab, ac\end{matrix};q,-bcx\right) = \frac{(cq^{-n}/a;q)_n}{(ac;q)_n}\,{}_3\varphi_2\left(\begin{matrix} q^{-n}, a^2q^n, -abx \\ ab, qa/c\end{matrix};q,q\right),
\tag{3.4}
$$

see [21, Prop. 5.3]. The big $q$-Jacobi polynomials are orthogonal with respect to a positive discrete measure supported on $-q^{\mathbb{Z}_{\geq 0}}\cup -q^{\mathbb{N}}/bc$ for $ab<1$, $ac<1$, $qa/b<1$, $qa/c<1$ and $bc<0$, see [21, §10]. See Andrews and Askey [1], or [7, §7.3], [14] for the standard definition of the big $q$-Jacobi polynomials.

**3.3. The big $q$-Bessel functions.** If we replace $c$ by $c\varepsilon$ and $\gamma$ by $\gamma\varepsilon$ in (3.2), we obtain the following limit

$$
\lim_{\varepsilon\downarrow 0}\phi_{\gamma\varepsilon}(x;a;b,c\varepsilon;q) = {}_1\varphi_1\left(\begin{matrix}-1/x\\ ab\end{matrix};q,-\frac{abcx}{\gamma}\right).
\tag{3.5}
$$

We define the big $q$-Bessel function by

$$
J_\gamma(x;a;q) = {}_1\varphi_1\left(\begin{matrix}-1/x\\ a\end{matrix};q,a\gamma x\right).
\tag{3.6}
$$

The big $q$-Bessel function is a solution to $LJ_\gamma(\cdot;a;q) = -\gamma J_\gamma(\cdot;a;q)$, with $L$ as in (2.1) with $A(x) = x^{-1}(1+1/ax)$ and $B(x) = q(1+1/x)/ax$. Taking the limit in the big $q$-Jacobi function transform pair (3.3) with $z$ fixed, shows that for $\gamma>0$, $0<a<1$,

(3.7)

$$
\int_{-1}^{\infty(q/a\gamma)}\big(J_{\gamma q^k}J_{\gamma q^l}\big)(x;a;q)\frac{(-qx;q)_\infty}{(-ax;q)_\infty}\,d_qx = \delta_{k,l}(1-q)\frac{(q;q)_\infty^2\theta(-a\gamma)}{(a;q)_\infty^2\theta(-\gamma)}a^{-k}(-q^k\gamma;q)_\infty
$$

for $k, l \in \mathbb{Z}$. Here the extra parameter $z$ in the measure of the big $q$-Jacobi function transform (3.3) is in the limit inverse proportional to $\gamma$. Under the limit transition (3.5) the only part of the spectrum of (3.3) that survives, is the infinite set of discrete mass points tending to $-\infty$. Moreover, the big $q$-Bessel functions $J_{\gamma q^l}(\cdot; a; q)$ form a complete orthogonal set in the weighted $L^2$-space for the measure in (3.7). See [6] for the proof of (3.7).

The big $q$-Bessel function in (3.5) can also be obtained by taking the limit from the big $q$-Jacobi polynomials and then the orthogonality relations (3.7) can be obtained from the orthogonality relations for the big $q$-Jacobi polynomials in a rigorous way, see [6, §6].

## 4. Little $q$-analogue of the Jacobi function scheme

In this section we consider the analogue of Figure 1.1 on the level of the little $q$-Jacobi case. So the second order difference operator $L$ is now as in (2.1) with

$$A(x) = a^2(1 + \frac{1}{ax}), \qquad B(x) = (1 + \frac{q}{bx}). \tag{4.1}$$

The general set of eigenfunctions for $L$ with $A$ and $B$ as in (4.1) is given by the solutions to second order $q$-difference equation that is the $q$-analogue of the hypergeometric differential equation, see [7, Ch. 1].

**4.1. The little $q$-Jacobi functions.** The little $q$-Jacobi functions are defined by

$$\phi_\gamma(x; a; b; q) = {}_2\varphi_1\left(\begin{matrix} a\gamma, a/\gamma \\ ab \end{matrix}; q, -bx\right), \tag{4.2}$$

and they satisfy $L\phi_\gamma(\cdot; a; b; q) = (-1 - a^2 + a(\gamma + \gamma^{-1}))\phi_\gamma(\cdot; a; b; q)$. For $y > 0$, $a > b > 0$, and $ab < 1$ we have the transform pair

$$\begin{aligned} \hat{u}(\gamma) &= \sum_{k=-\infty}^{\infty} u(k)\phi_\gamma(yq^k; a; b; q)\, a^{2k} \frac{(-q^{1-k}/ay; q)_\infty}{(-q^{1-k}/by; q)_\infty}, \\ u(k) &= C \int_{\mathbb{C}^*} \hat{u}(\gamma)\phi_\gamma(yq^k; a; b; q)\, d\nu(\gamma; a; b, aby; q/aby|q, -1), \end{aligned} \tag{4.3}$$

for $C = (ab;q)_\infty^2 \theta(-by)^2 / K(a; b, aby; q/aby; -1)$ and using the notation (2.7). Observe that cancellation in the weight $\Delta$ (2.6) occurs in (4.3), since $cd = q$. For a proof of (4.3) see Kakehi [12] or [22, App. A], where the result is given for more general parameter values.

**4.2. The little $q$-Jacobi polynomials.** The little $q$-Jacobi polynomials are the polynomial eigenfunctions of $L$ as in (2.1), (4.1), and they occur for $\gamma = aq^n$, $n \in \mathbb{Z}_{\geq 0}$;

$$\phi_{aq^n}(x; a; b; q) = {}_2\varphi_1\left(\begin{matrix} q^{-n}, a^2q^n \\ ab \end{matrix}; q, -bx\right). \tag{4.4}$$

This is not the standard expression for the little $q$-Jacobi polynomials as introduced by Andrews and Askey [1], or see [7, §7.3], [14], where the orthogonality relations can be found.

**4.3. The little $q$-Bessel functions.** If we replace $\gamma$ by $\gamma\varepsilon$ and $x$ by $x\varepsilon$, we obtain the little $q$-Bessel functions from the little $q$-Jacobi functions;

$$\lim_{\varepsilon\downarrow 0}\phi_{\gamma\varepsilon}(x\varepsilon;a;b;q)={}_1\varphi_1\left(\begin{matrix}0\\ ab\end{matrix};q,-\frac{abx}{\gamma}\right). \tag{4.5}$$

The little $q$-Bessel function is defined by

$$j_\gamma(x;a;q)={}_1\varphi_1(0;a;q,q\gamma x). \tag{4.6}$$

Note that the little $q$-Bessel function is self-dual; $j_\gamma(x;a;q)=j_x(\gamma;a;q)$. These $q$-analogues of the Bessel function are also known under the name ${}_1\varphi_1$ $q$-Bessel function or Hahn-Exton $q$-Bessel function, see [28] and [20] for historic references to Hahn, Exton and Jackson. The little $q$-Bessel functions are eigenfunctions of $Lj_\gamma(\cdot;a;q)=-q\gamma j_\gamma(\cdot;a;q)$ with $L$ as in (2.1) with $A(x)=a/x$ and $B(x)=q/x$. This is the second order $q$-difference equation for the ${}_1\varphi_1$-series, and we obtain this from the second order $q$-difference equation for the little $q$-Jacobi function. The little $q$-Bessel functions satisfy the orthogonality relations for $0<a<1$, see Koornwinder and Swarttouw [28, Prop. 2.6],

$$\sum_{k=-\infty}^{\infty}a^k j_{q^n}(q^k;a;q)\,j_{q^m}(q^k;a;q)=\delta_{n,m}a^{-n}\frac{(q;q)_\infty^2}{(a;q)_\infty^2},\qquad n,m\in\mathbb{Z}. \tag{4.7}$$

In the limit transition (4.5) of the little $q$-Jacobi function transform (4.3) the only part of the spectrum that survives the contraction is the infinite set of discrete mass points tending to $-\infty$, which leads to a formal derivation of (4.7). We note that the extra degree of freedom in the measure of the little $q$-Jacobi function transform (4.3) drops out in the limit. By self-duality we see that the little $q$-Bessel functions form a complete orthogonal set with respect to the discrete measure in (4.7).

The same limit transition of the little $q$-Jacobi polynomials to the little $q$-Bessel functions as in (4.5) is valid, and in the limit the orthogonality relations for the little $q$-Jacobi polynomials tend to the orthogonality relations (4.7), see [28] for a rigorous proof.

## 5. Limit transitions

In the previous sections limits within each level of the $q$-analogue of the Jacobi function schemes have been discussed. Now there are also limits from the Askey-Wilson polynomials to the big $q$-Jacobi polynomials and from the big $q$-Jacobi polynomials to the little $q$-Jacobi polynomials, see [26], [34] or [14]. In this section we show that these limit transitions also hold for the appropriate analogues of the Jacobi and Bessel function.

**5.1. Limit from the Askey-Wilson case to the big $q$-case.** In the Askey-Wilson function of (2.3) we replace $(a,b,c,d,x)$ by $(a/\varepsilon,b\varepsilon,c\varepsilon,d/\varepsilon,-x/\varepsilon)$. Then, using (2.4), the Askey-Wilson function tends to the big $q$-Jacobi function (3.2) as $\varepsilon\downarrow 0$;

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\phi_\gamma(-\frac{x}{\varepsilon};\frac{a}{\varepsilon};b\varepsilon,c\varepsilon,\frac{d}{\varepsilon}|q)&=\frac{(-q\tilde{a}x\gamma/d,\tilde{c}/\gamma;q)_\infty}{(\gamma\tilde{c},ac,qa/d,-qx/d;q)_\infty}\,{}_3\varphi_2\left(\begin{matrix}-bx,\tilde{a}\gamma,\tilde{b}\gamma\\ ab,-q\tilde{a}\gamma x/d\end{matrix};q,\frac{\tilde{c}}{\gamma}\right)\\&=\frac{1}{(qa/d;q)_\infty}\phi_\gamma(\frac{x}{a};\tilde{a};\tilde{b},\tilde{c};q),\end{aligned} \tag{5.1}$$

where we have used [7, (III.9)] in the second equality. Keeping $t$ fixed we can formally take the limit transition within the Askey-Wilson function transform pair (2.9) to recover the big $q$-Jacobi function transform (3.3) with parameters $(a, b, c, z)$ replaced by $(\tilde{a}, \tilde{b}, \tilde{c}, -td/a)$. Taking $\gamma = \tilde{a}q^n$, $n \in \mathbb{Z}_{\geq 0}$, gives back the limit transition from the Askey-Wilson polynomials to the big $q$-Jacobi polynomials.

In the Askey-Wilson $q$-Bessel function we replace $(a, b, x, \gamma)$ by $(a/\varepsilon, b\varepsilon, -x/\varepsilon, \gamma\varepsilon)$ in (2.12) and take the limit $\varepsilon \downarrow 0$, which gives the big $q$-Bessel function;

$$\lim_{\varepsilon \downarrow 0} J_{\gamma\varepsilon}(-\frac{x}{\varepsilon}; \frac{a}{\varepsilon}; b\varepsilon|q) = J_{q\gamma/b}(\frac{x}{a}; ab; q). \tag{5.2}$$

In this limit transition the second order $q$-difference equation for the Askey-Wilson $q$-Bessel function goes over into the second order $q$-difference equation for the big $q$-Bessel function. And the orthogonality relations (2.14) go over into the orthogonality relations (3.7), since only the discrete mass points of the measure survive in the limit.

**5.2. Limit from the big $q$-case to the little $q$-case.** The big $q$-Jacobi function of (3.2) tends to the little $q$-Jacobi function in (4.2) by

$$\lim_{c \downarrow 0} \phi_\gamma(\frac{x}{c}; a; b, c; q) = \phi_\gamma(x; a; b; q). \tag{5.3}$$

In this limit transition the big $q$-Jacobi function transform (3.3) tends to the little $q$-Jacobi function after taking $z = y/c$ for the extra parameter in the big $q$-Jacobi function transform (3.3). The second order $q$-difference equation for the big $q$-Jacobi functions tends to the second order $q$-difference equation for the little $q$-Jacobi functions under (5.3).

In the big $q$-Bessel function (3.6) we replace $x$ by $x/c$ and $\gamma$ by $cq\gamma/a$ and take the limit

$$\lim_{c \downarrow 0} J_{cq\gamma/a}(\frac{x}{c}; a; q) = j_\gamma(x; a; q) \tag{5.4}$$

to find the little $q$-Bessel function (4.6). In this limit transition the big $q$-Hankel orthogonality relations (3.7) tend to the little $q$-Hankel orthogonality relations (4.7), for which we observe that the $\gamma$-dependence drops out in the limit. The second order $q$-difference equation for the big $q$-Bessel functions tends to the second order $q$-difference equation for the little $q$-Bessel functions under (5.4).

## 6. Duality and factoring of limits

As already observed, the Askey-Wilson function transform and the little $q$-Hankel transform are self-dual. However, the transforms in between are not self-dual, and the dual transforms are described in this section. It turns out that these are related to indeterminate moment problems, see [6], [21]. Moreover, we find that the limit transition of the little $q$-Jacobi function to the little $q$-Hankel transform and of the Askey-Wilson function to the Askey-Wilson $q$-Hankel transform factors through the transforms dual to the big $q$-Hankel transform and dual to the big $q$-Jacobi function transform.

**6.1. Duality between the little $q$-Jacobi function transform and the Askey-Wilson $q$-Hankel transform.** The little $q$-Jacobi function of (4.2) and the Askey-Wilson $q$-Bessel function of (2.12) are related by interchanging the geometric parameter $x$ and the spectral parameter $\gamma$;

$$J_\gamma(x;a;b|q) = \phi_x(\frac{q\gamma}{ab};a,b;q). \tag{6.1}$$

The orthogonality relations (2.14) and the transform (4.3) can be obtained from each other by using (6.1). So the little $q$-Jacobi function and the Askey-Wilson $q$-Bessel function are also eigenfunctions of a second order $q$-difference equation acting on the spectral parameter.

**6.2. The dual to the big $q$-Hankel transform.** The big $q$-Bessel functions form an orthogonal basis for the weighted $L^2$-space on $[-1,\infty(q/a\gamma))_q$ described in (3.7), see [6]. The corresponding dual orthogonality relations are then labeled by the support of this measure, i.e. by $-q^{\mathbb{Z}_{\geq 0}}$ and by $q^{\mathbb{Z}}/a\gamma$. In the first case, the big $q$-Bessel functions at $x=-q^n$ are related to the $q$-Laguerre polynomials in $\gamma$;

$$J_\gamma(-q^n;a;q) = {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ a \end{matrix};q,-a\gamma q^n\right), \tag{6.2}$$

and the dual orthogonality relations for $J_\gamma(-q^n;a;q)$ and $J_\gamma(-q^m;a;q)$, $n,m\in\mathbb{Z}_{\geq 0}$, reduce to the orthogonality relations for the $q$-Laguerre polynomials related to Ramanujan's ${}_1\psi_1$-sum, see Moak [30, Thm. 2], or [7, Exer. 7.43(ii)]. The big $q$-Bessel functions are eigenfunctions of a second order $q$-difference operator in the spectral parameter $\gamma$;

$$\Big((1+\frac{1}{\gamma})(T_q^\gamma - 1) + \frac{q}{a\gamma}(T_{q^{-1}}^\gamma - 1)\Big)J_\gamma(x;a;q) = -(1+x)J_\gamma(x;a;q), \tag{6.3}$$

where $(T^\gamma_{q^{\pm1}}f)(\gamma) = f(q^{\pm1}\gamma)$. Note that (6.3) is nothing but the second order $q$-difference equation for the ${}_1\varphi_1$-series.

It is known that the $q$-Laguerre polynomials correspond to an indeterminate moment problem and that this solution to the moment problem is not extremal in the sense of Nevannlina, meaning that the $q$-Laguerre polynomials are not dense in the corresponding weighted $L^2$-space. The functions of $\gamma$ defined by, $p\in\mathbb{Z}$,

$$J_\gamma(\frac{q^{p+1}}{a\gamma};a;q) = {}_1\varphi_1\left(\begin{matrix} -a\gamma q^{-1-p} \\ a \end{matrix};q,q^{p+1}\right) = \frac{(q^{p+1};q)_\infty}{(a;q)_\infty}\,{}_1\varphi_1\left(\begin{matrix} -\gamma \\ q^{p+1} \end{matrix};q,a\right) \tag{6.4}$$

display $q$-Bessel coefficient behaviour. Here we used a limit case of the transformation formula [7, (III.2)]. These $q$-Bessel coefficients complement the orthogonal set of $q$-Laguerre polynomials into an orthogonal basis of the corresponding weighted $L^2$-space. This is a direct consequence of of the orthogonality relations (3.7) and the completeness. See [6] for details, where also the spectral analysis of (6.3) is given.

**6.3. The dual to the big $q$-Jacobi function transform.** Evaluating the big $q$-Jacobi function at the point $-q^k$, $k \in \mathbb{Z}_{\geq 0}$, gives a terminating series in which the base is inverted to $q^{-1}$;

$$\phi_\gamma(-q^k; a; b, c; q) = {}_3\varphi_2\left(\begin{matrix} q^k, \gamma/a, 1/a\gamma \\ 1/ab, 1/ac \end{matrix}; q^{-1}, q^{-1}\right). \tag{6.5}$$

The right hand side is a polynomial of degree $k$ in $\frac{1}{2}(\gamma+\gamma^{-1})$, which is a continuous dual $q^{-1}$-Hahn polynomial with parameters $a^{-1}$, $b^{-1}$, $c^{-1}$, i.e. an Askey-Wilson polynomial with one parameter set to zero, see [7, §7.5], [14]. Hence, the transform (3.3) gives us an orthogonality measure for the continuous dual $q^{-1}$-Hahn polynomials, together with a complementing set of orthogonal functions in $\frac{1}{2}(\gamma+\gamma^{-1})$, namely $\phi_\gamma(zq^l; a; b, c; q)$, $l \in \mathbb{Z}$. See [21, §9] for more details. In particular we find that the big $q$-Jacobi functions are eigenfunctions to a second order $q$-difference operator in the spectral parameter $\gamma$, see [11];

$$\begin{gathered}\big(A(\gamma)(T_q^\gamma - 1) + A(\gamma^{-1})(T_{q^{-1}}^\gamma - 1)\big)\phi_\gamma(x; a; b, c; q) = -(1+x)\phi_\gamma(x; a; b, c; q), \\ A(\gamma) = \frac{(1-1/\gamma a)(1-1/\gamma b)(1-1/\gamma c)}{(1-\gamma^{-2})(1-1/\gamma^2 q)},\end{gathered} \tag{6.6}$$

cf. the second order $q$-difference equation for the continuous dual $q^{-1}$-Hahn polynomials [2], [14]. See also Rosengren [33, §4.6] for another orthogonality measure for the continuous dual $q^{-1}$-Hahn polynomials.

**6.4. Remaining limit transitions.** Using duality and the limit transitions in §5 we obtain the remaining limit transitions in Figure 1.2, namely the limits from the Askey-Wilson functions to the continuous dual $q^{-1}$-Hahn polynomials and the associated $q$-Bessel functions, and from this family to the Askey-Wilson $q$-Bessel functions, and from the little $q$-Jacobi functions to the $q$-Laguerre polynomials and the associated $q$-Bessel functions, and from this family to the little $q$-Bessel functions. These limits are formal and can be taken in the second order $q$-difference equation and in the transform pairs.

As an example, we illustrate the limit from the Askey-Wilson functions to the continuous dual $q^{-1}$-Hahn polynomials and the associated $q$-Bessel functions. Using the duality (2.5) we have for $x_k = -aq^k$, $k \in \mathbb{Z}_{\geq 0}$, and $\varepsilon > 0$ sufficiently small,

$$\phi_{-x_k/\varepsilon}(\gamma; \tilde{a}; \tilde{b}, \tilde{c}; \frac{\tilde{d}}{\varepsilon^2}|q) = \phi_\gamma(-\frac{x_k}{\varepsilon}; \frac{a}{\varepsilon}; b\varepsilon, c\varepsilon; \frac{d}{\varepsilon}|q). \tag{6.7}$$

By (5.1) the limit $\varepsilon \downarrow 0$ gives

$$\lim_{\varepsilon\downarrow 0} \phi_{-x_k/\varepsilon}(\gamma; \tilde{a}; \tilde{b}, \tilde{c}; \frac{\tilde{d}}{\varepsilon^2}|q) = \frac{1}{(qa/d; q)_\infty}\phi_\gamma(-q^k; \tilde{a}; \tilde{b}, \tilde{c}; q) \tag{6.8}$$

and the right hand side is a continuous dual $q^{-1}$-Hahn polynomial of degree $k$ up to a constant, see (6.5). Replacing $x_k$ by the discrete weights $y_l = -tdq^l$, $l \in \mathbb{Z}$, in (6.7), (6.8) we see that $\phi_{-y_l/\varepsilon}(\gamma; \tilde{a}; \tilde{b}, \tilde{c}; \frac{\tilde{d}}{\varepsilon^2}|q)$ tend to $q$-Bessel coefficient type functions which complement

the continuous dual $q^{-1}$-Hahn polynomials to an orthogonal basis in the corresponding weighted $L^2$-space. The non-extremal measure, which is parametrised by the $z$-parameter in the big $q$-Jacobi function transform (3.3), corresponds to $z = -td/a$. In this limit the Askey-Wilson function transform formally tends to the the orthogonality relations for the continuous dual $q^{-1}$-Hahn polynomials and the corresponding $q$-Bessel functions. For this we only need to remark that, by duality, this reduces to limit transition (5.1) of the Askey-Wilson function transform to the big $q$-Jacobi function transform.

All the other cases can be considered in a similar manner and for completeness we give the underlying limit transitions;

$$\lim_{c\downarrow 0} \phi_\gamma(\frac{x}{c}; a; b, c; q) = J_{abx/q}(\gamma; a; b; q), \tag{6.9}$$

$$\lim_{\varepsilon\downarrow 0} \phi_{\gamma/\varepsilon}(\varepsilon x; \frac{a}{\varepsilon}; \varepsilon b; q) = J_{ax}(-\frac{\gamma}{a}; ab; q), \tag{6.10}$$

$$\lim_{\varepsilon\downarrow 0} J_{\varepsilon\gamma}(\frac{x}{\varepsilon}; a; q) = j_x(\frac{\gamma a}{q}; a; q). \tag{6.11}$$

Let us finally note that this makes the scheme of limit transitions in Figure 1.2 into a commutative diagram.

## 7. Concluding remarks

**7.1. Quantum group theoretic interpretation.** There is a group theoretic interpretation for the scheme of Figure 1.1, see Koornwinder [25]. Here the Jacobi polynomials have an interpretation on the compact real Lie group $SU(2)$ as matrix elements of finite-dimensional irreducible unitary representations, whereas the Jacobi functions have an interpretation as matrix elements of infinite-dimensional irreducible unitary representations of the non-compact real Lie group $SU(1,1)$. The real Lie groups $SU(2)$ and $SU(1,1)$ are both real forms of the complex Lie group $SL(2,\mathbb{C})$. Bessel functions occur as matrix elements of infinite-dimensional irreducible unitary representations of the group $E(2)$ of motions of the Euclidean plane, and then the limit transition can be interpreted on the level of Lie groups as a contraction.

There is also a quantum group theoretic interpretation for the Askey-Wilson function scheme in Figure 1.2. Here the Askey-Wilson, big and little $q$-Jacobi polynomials are interpreted as matrix elements of irreducible unitary representations of the quantum $SU(2)$ group, see e.g. [19], [26], [31], [32] and further references. These interpretations also naturally lead to the limit transitions from Askey-Wilson polynomials to big $q$-Jacobi polynomials, and from big $q$-Jacobi polynomials to little $q$-Jacobi polynomials, see Koornwinder [26]. The various $q$-Hankel transforms and $q$-Bessel functions have an interpretation on the quantum $E(2)$ group (or better, its twofold covering); see [16], [37] for the little $q$-Bessel function; see [3], [18] for the big $q$-Bessel function, and see [15], [17] for the Askey-Wilson $q$-Bessel function. The limit transitions from the $q$-analogues of the Jacobi polynomials to the $q$-analogues of the Bessel functions as described in §§2,3, 4 are motivated from a similar contraction from the quantum $SU(2)$ group to the quantum $E(2)$ group. The quantum $SU(2)$ group is a real form of the quantum $SL(2,\mathbb{C})$ group, and one of the other real forms is the quantum $SU(1,1)$ group. The interpretation of the little $q$-Jacobi functions

on the quantum $SU(1,1)$ group as matrix elements of irreducible unitary representations is due to Masuda et al. [29], Kakehi [12], Kakehi, Masuda and Ueno [13] and Vaksman and Korogodskiĭ [38]. For the big $q$-Jacobi and Askey-Wilson function such an interpretation is given in [22]. This interpretation leads in a natural way to the limit transitions of the Askey-Wilson functions to big $q$-Jacobi functions, and from big $q$-Jacobi functions to little $q$-Jacobi functions as described in §5. Moreover, from a contraction procedure from the quantum $SU(1,1)$ group to the quantum $E(2)$ group we obtain the limit transition of the $q$-analogues of the Jacobi function to the corresponding $q$-analogues of the Bessel function, see §§2, 3 and 4. Due to these representation theoretic interpretation of the three types of $q$-special functions on the quantum $SU(1,1)$, $E(2)$ and $SU(2)$ groups, we may view them as $q$-analogues of the Jacobi functions, Bessel functions and Jacobi polynomials, respectively. This explains the naming of the $q$-special functions in Figure 1.2.

The second order difference equation for the $q$-special functions follows from the action of the Casimir element for these quantum groups. In §6.1 we have seen that the Askey-Wilson $q$-Bessel functions and the little $q$-Jacobi functions both satisfy a second order $q$-difference equation in the spectral parameter. The spectral parameter corresponds to the representation label of the irreducible representations of the quantum $E(2)$ group and the quantum $SU(1,1)$ group respectively. The second order $q$-difference equation in the spectral parameter can then be obtained from the tensor product decomposition of a three-dimensional (non-unitary) representation with an irreducible infinite dimensional representation into three irreducible infinite dimensional representations of the corrresponding quantum groups, see e.g. [19, Remark 7.2] for the corresponding statement for the quantum $SU(2)$ group.

**7.2. Further extensions.** We briefly sketch some possible directions related to the Askey-Wilson function scheme.

**1.** From the previous subsection we see that the scheme depicted in Figure 1.2 is motivated by the simplest quantum groups; namely for $SU(2)$, $SU(1,1)$, and $E(2)$. We would expect that a greater extension of the scheme in Figure 1.2 is possible using more complicated quantum groups, especially higher rank quantum groups, or other interpretations of special functions on quantum groups, such as Clebsch-Gordan, Racah or other type of overlap coefficients. See e.g. Koornwinder [25] for a further extension of the Jacobi function scheme in Figure 1.1. In such an extension of Figure 1.2 the other $q$-analogues of the Bessel function, as studied by Ismail [9], should also find a place.

**2.** For the Hankel transform and Jacobi function transform there are systems of orthogonal polynomials that are mapped onto each other, such as the Laguerre polynomials for the Hankel transform or the Jacobi polynomials that are mapped onto the Wilson polynomials by the Jacobi function transform, see [25]. It would be interesting to know what the corresponding results for the $q$-analogues of these transforms are.

## References

1. G.E. Andrews and R. Askey, *Classical orthogonal polynomials*, "Polynômes Orthogonaux et Applications", LNM, vol. 1171, Springer Verlag, 1985, pp. 36–62.
2. R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **54** (1985), no. 319.

3. F. Bonechi, N. Ciccoli, R. Giachetti, E. Sorace, and M. Tarlini, *Free q-Schrödinger equation from homogeneous spaces of the 2-dim Euclidean quantum group*, Comm. Math. Phys. **175** (1996), 161–176.
4. B.M. Brown, W.D. Evans and M.E.H. Ismail, *The Askey-Wilson polynomials and q-Sturm-Liouville problems*, Math. Proc. Cambridge Philos. Soc. **119** (1996), 1–16.
5. J. Bustoz and S.K. Suslov, *Basic analog of Fourier series on a q-quadratic grid*, Methods Appl. Anal. **5** (1998), 1–38.
6. N. Ciccoli, E. Koelink and T.H. Koornwinder, *q-Laguerre polynomials and big q-Bessel functions and their orthogonality relations*, Methods Appl. Anal. **6** (1999), 109–127.
7. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge Univ. Press, 1990.
8. D.P. Gupta, M.E.H. Ismail and D.R. Masson, *Contiguous relations, basic hypergeometric functions, and orthogonal polynomials III. Associated continuous dual q-Hahn polynomials*, J. Comput. Appl. Math. **68** (1996), 115–149.
9. M.E.H. Ismail, *The zeros of basic Bessel functions, the function $J_{\nu+ax}(x)$, and associated orthogonal polynomials*, J. Math. Anal. Appl. **86** (1982), 1–19.
10. M.E.H. Ismail, D.R. Masson and S.K. Suslov, *The q-Bessel function on a q-quadratic grid*, "Algebraic Methods and $q$-Special Functions" (J.F. van Diejen and L. Vinet, eds.), CRM Proc. Lect. Notes, vol. 22, AMS, 1999, pp. 183–200.
11. M.E.H. Ismail and M. Rahman, *The associated Askey-Wilson polynomials*, Trans. Amer. Math. Soc. **328** (1991), 201–237.
12. T. Kakehi, *Eigenfunction expansion associated with the Casimir operator on the quantum group* $SU_q(1,1)$, Duke Math. J. **80** (1995), 535–573.
13. T. Kakehi, T. Masuda and K. Ueno, *Spectral analysis of a q-difference operator which arises from the quantum $SU(1,1)$ group*, J. Operator Theory **33** (1995), 159–196.
14. R. Koekoek and R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Delft University of Technology Report no. 98-17 (1998).
15. H.T. Koelink, *On quantum groups and q-special functions*, dissertation, Rijksuniversiteit Leiden, 1991.
16. H.T. Koelink, *The quantum group of plane motions and the Hahn-Exton q-Bessel function*, Duke Math. J. **76** (1994), 483–508.
17. H.T. Koelink, *The quantum group of plane motions and basic Bessel functions*, Indag. Math. (N.S.) **6** (1995), 197–211.
18. H.T. Koelink, *Yet another basic analogue of Graf's addition formula*, J. Comput. Appl. Math. **68** (1996), 209–220.
19. H.T. Koelink, *Askey-Wilson polynomials and the quantum $SU(2)$ group: survey and applications*, Acta Appl. Math. **44** (1996), 295–352.
20. H.T. Koelink, *Some basic Lommel polynomials*, J. Approx. Theory **96** (1999), 345–365.
21. E. Koelink and J.V. Stokman, *The big q-Jacobi function transform*, preprint 40 p., `math.CA/9904111` (1999).
22. E. Koelink and J.V. Stokman, with an appendix by M. Rahman, *Fourier transforms on the quantum $SU(1,1)$ group*, preprint 77 p., `math.QA/9911163` (1999).
23. E. Koelink and J.V. Stokman, *The Askey-Wilson function transform*, preprint (to appear).
24. T.H. Koornwinder, *Jacobi functions and analysis on noncompact semisimple Lie groups*, "Special Functions: Group Theoretical Aspects and Applications" (R. A. Askey, T. H. Koornwinder and W. Schempp, eds.), Reidel, 1984, pp. 1–85.
25. T.H. Koornwinder, *Group theoretic interpretations of Askey's scheme of hypergeometric orthogonal polynomials*, "Orthogonal Polynomials and their Applications", LNM, vol. 1329, Springer, 1988, pp. 46–72.
26. T.H. Koornwinder, *Askey-Wilson polynomials as zonal spherical functions on the $SU(2)$ quantum group*, SIAM J. Math. Anal. **24** (1993), 795–813.
27. T.H. Koornwinder, *Compact quantum groups and q-special functions*, "Representations of Lie Groups and Quantum Groups", Pitman Res. Notes Math. Ser., vol. 311, Longman Sci. Tech., 1994, pp. 46–128.
28. T.H. Koornwinder and R.F. Swarttouw, *On q-analogues of the Fourier and Hankel transforms*, Trans. Amer. Math. Soc. **333** (1992), 445–461.

29. T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi, Y. Saburi and K. Ueno, *Unitary representations of the quantum group $SU_q(1,1)$: Structure of the dual space of $U_q(sl(2))$*, Lett. Math. Phys. **19** (1990), 197–194; *II: Matrix elements of unitary representations and the basic hypergeometric functions*, 195–204.
30. D.S. Moak, *The q-analogue of the Laguerre polynomials*, J. Math. Anal. Appl. **81** (1981), 20–47.
31. M. Noumi, *Quantum groups and q-orthogonal polynomials. Towards a realization of Askey-Wilson polynomials on $SU_q(2)$*, "Special Functions" (M. Kashiwara and T. Miwa, eds.), ICM-90 Satellite Conference Proceedings, Springer-Verlag, 1991, pp. 260–288.
32. M. Noumi and K. Mimachi, *Askey-Wilson polynomials as spherical functions on $SU_q(2)$*, "Quantum Groups", LNM, vol. 1510, Springer Verlag, 1992, pp. 98–103.
33. H. Rosengren, *A new quantum algebraic interpretation of the Askey-Wilson polynomials*, Contemp. Math. (to appear).
34. J.V. Stokman and T.H. Koornwinder, *On some limit cases of Askey-Wilson polynomials*, J. Approx. Theory **95** (1998), 310–330.
35. S.K. Suslov, *Some orthogonal very well poised ${}_8\phi_7$-functions*, J. Phys. A **30** (1997), 5877–5885.
36. S.K. Suslov, *Some orthogonal very-well-poised ${}_8\varphi_7$-functions that generalize Askey-Wilson polynomials*, preprint (1997).
37. L.L. Vaksman and L.I. Korogodskiĭ, *The algebra of bounded functions on the quantum group of motions of the plane and q-analogues of Bessel functions*, Soviet Math. Dokl. **39** (1989), 173–177.
38. L.L. Vaksman and L.I. Korogodskiĭ, *Spherical functions on the quantum group $SU(1,1)$ and a q-analogue of the Mehler-Fock formula*, Funct. Anal. Appl. **25** (1991), 48–49.
39. G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, 1944.

Technische Universiteit Delft, Faculteit Informatietechnologie en Systemen, Afd. Toegepaste Wiskundige Analyse, Postbus 5031, 2600 GA Delft, the Netherlands
*E-mail address*: `koelink@twi.tudelft.nl`

Centre de Mathématiques de Jussieu, Université Paris 6 Pierre et Marie Curie, 4, place Jussieu, F-75252 Paris Cedex 05, France
*E-mail address*: `stokman@math.jussieu.fr`